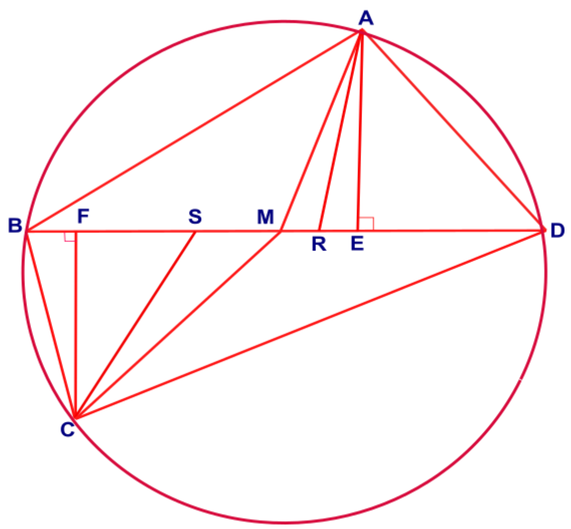
Author's Solution for 01.04.2023 Cash Award Math Rider

Given :

AE, AR & AM are respectively the altitude, angle bisector and median of ΔABD and CF, CS & CM are respectively the altitude, angle bisector & median of ΔCBD.   AE = CF.



To Prove: $(MS^2 - MR^2) = (FS^2 - ER^2)$

Before solving the problem, let us prove two results, (regarding Altitude and Angle-bisector) used in the solution of the problem. (See the box below)

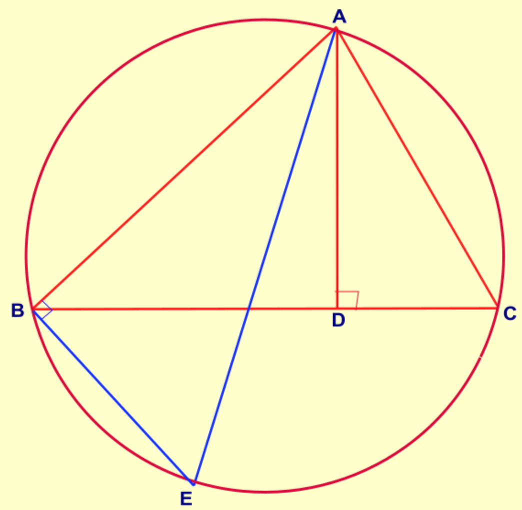
1. Result about Altitude:

An altitude of a triangle is always equal to the product of the sides of its vertex divided by the diameter of its circumcircle. (See the picture below)

Given :

In ΔABC, AD is the altitude from vertex A.

AE is the diameter of its circumcircle.



To prove:

$$AD = \frac{AB \times AC}{AE} = \frac{AB \times AC}{Diameter}$$

Proof :

Join BE.

Now, $\angle ADC = \angle ABE = 90°$ $(\because AE \text{ is diameter})$

$\angle ACD = \angle AEB$ (Angles in same segment)

$\therefore \Delta ADC \sim \Delta ABE$

$$\Rightarrow \frac{AD}{AB} = \frac{AC}{AE}$$

$$\Rightarrow AD = \frac{AB \times AC}{AE} = \frac{AB \times AC}{Diameter}$$ -------------------- **Proved**

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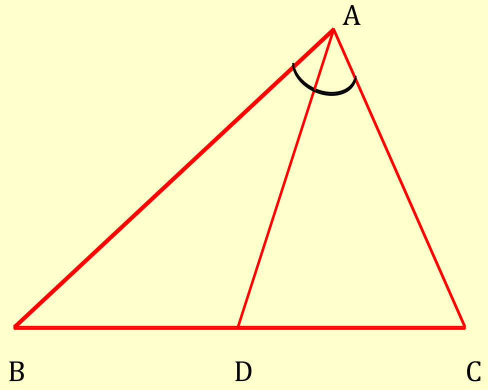
2. Result about Angle -Bisector:

The square of an Angle bisector of a triangle is always equal to the difference between product of the sides of its vertex and the product of the line segments of the opposite side created by it. (See the picture below)



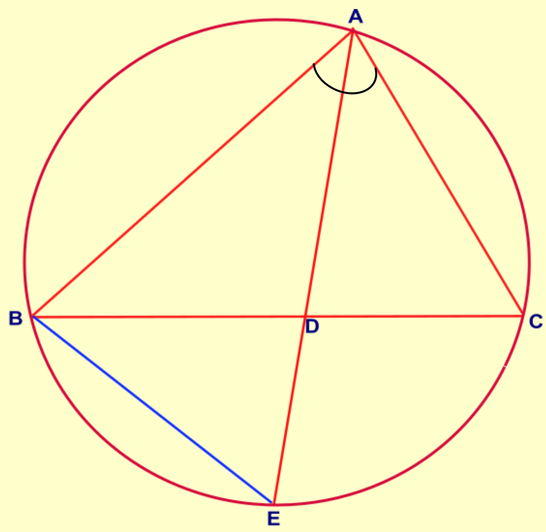
Given: In ΔABC, AD is the bisector of $\angle BAC$.

To Prove: $AD^2 = AB \times AC - BD \times DC$

Construction:

Draw the circumcircle of ΔABC

Produce AD to meet the circumcircle at E. Join BE.



Proof :

$\angle AEB = \angle ACD$ (Angles in same segment)

$\angle BAE = \angle CAD$ $(Given)$

$\therefore \Delta ABE \sim \Delta ADC$

$\Rightarrow \frac{AB}{AD} = \frac{AE}{AC}$

$\Rightarrow AE \times AD = AB \times AC$

AD (AD+DE) = AB x AC

$\Rightarrow AD^2 = AB \times AC - AD \times DE$

$\Rightarrow AD^2 = AB \times AC - BD \times DC$ $[\because AD \times DE = BD \times DC]$ ------------------ **Proved**

As per the result (1) above (Result about altitude),(See box above)

$AE = \frac{AB \times AD}{Diameter}$ ------------------------ (1)

And CF = $\frac{CB \times CD}{Diameter}$ -------------------(2)

$AE = CF$ (Given) ----------------- (3)

(1), (2) & (3) → AB x AD = CB x CD ----------- (4)

As per result (2) above (Result about Angle bisector), (See box above)

AR^2 = AB x AD - BR x RD ------------------(5)

And CS^2 = CB x CD - BS x SD ------------(6)

(6) - (5) $\Longrightarrow$

$CS^2 - AR^2 = (\text{CB x CD} - \text{BS x SD}) - (\text{AB x AD} - \text{BR x RD})$

$(CF^2 + FS^2) - (AE^2 + ER^2) = \text{CB x CD - BS x SD} - \text{AB x AD} + \text{BR x RD}$

$AE^2 + FS^2 - AE^2 - ER^2 = \text{BR x RD} - \text{BS x SD}$

[given AE=CF and AB x AD = CB x CD vide (4) above]

$FS^2 - ER^2 = \text{(BM + MR) x (BM-MR) - (BM - MS) x (BM+MS)}$ [as M is the midpoint of BD]

$FS^2 - ER^2 = (BM^2 - MR^2) - (BM^2 - MS^2)$

$FS^2 - ER^2 = MS^2 - MR^2$ ---------------------------- Proved

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